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This essay discusses some geometric effects associated with gravitomagnetic fields and gravitomagnetic charge as well as the gravity theory of the latter. Gravitomagnetic charge is the duality of gravitoelectric charge (mass) and is therefore also termed the *dual mass* which represents the topological property of gravitation. The field equation of gravitomagnetic matter is suggested and a static spherically symmetric solution of this equation is offered. A possible explanation of the anomalous acceleration acting on Pioneer spacecrafts are briefly proposed.

KEY WORDS: Gravitational analogue, gravitomagnetic charge, field equation

I. INTRODUCTION

Considering the following gravitational analogues of electromagnetic phenomena is of physical interest: (1) in electrodynamics a charged particle is acted upon by the Lorentz magnetic force and, in the similar fashion, a particle is also acted upon by the gravitational Lorentz force in weak-gravity theory [1,2]. According to the principle of equivalence, further analysis shows that in the non-inertial rotating reference frame, this gravitational Lorentz force is just the fictitious Coriolis force; (2) there exists Aharonov-Bohm effect in electrodynamics [3], accordingly, the so-called gravitational Aharonov-Bohm effect, *i.e.*, the gravitational analogue of Aharonov-Bohm effect also exists in the theory of gravitation, which is now termed the Aharonov-Carmi effect [4–6]; (3) a particle with intrinsic spin possesses a gravitomagnetic moment [7] of such magnitude that it equals the spin of this particle. The interaction of spinning gravitomagnetic moment with the gravitomagnetic field is called spin-gravity coupling, which is similar to the interaction between spinning magnetic moment and magnetic fields in electrodynamics; (4) in the rotating reference frame, the rotating frequency relative to the fixing frame may be considered the effective gravitomagnetic field strength that is independent of the Newtonian gravitational constant, G , in accordance with the principle of equivalence. This, therefore, means that the nature of spin-rotation coupling [8,9] is the interaction of spinning gravitomagnetic moment with gravitomagnetic fields; (5) it is well known that geometric phase reflecting the global and topological properties of evolution of the quantum systems [10,11] appears in systems whose Hamiltonian is time-dependent or possessing evolution parameters. Apparently, the geometric phase in the Aharonov-Bohm effect and Aharonov-Carmi effect results from the presence of the evolution parameter in the Hamiltonian. We suggested another geometric phase [1] that exists in the spin-rotation coupling system where the rotating frequency of the rotating frame is time-dependent. Investigation of this geometric phase is believed to be a potential application to the Earth's time-dependent rotating frequency (frequency fluctuations), namely, by measuring the geometric phase difference of spin polarized vertically down and up in the neutron-gravity interferometry experiment, one may obtain the information concerning the variation of the Earth's rotation.

For the present, it is possible to investigate quantum mechanics in weak-gravitational fields [12,13], with the development of detecting and measuring technology, particularly laser-interferometer technology, low-temperature technology, electronic technology and so on. These investigations enable physicists to test validity or universality of fundamental laws and principles of general relativity. It should be noted that the Aharonov-Carmi effect and the geometric phase factor in the time-dependent spin-rotation coupling reflect the aspects of geometric properties in gravity. Both of them are related to the gravitomagnetic fields. In this essay, the author discusses another geometric or topological aspect of the gravity, *i.e.*, the gravitomagnetic charge that is the gravitational analogue of magnetic monopole in electrodynamics.

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In electrodynamics, electric charge is a Noether charge while its dual charge (magnetic charge) is a topological charge, since the latter relates to the singularity of non-analytical vector potentials. Magnetic monopole [14] attracts attentions of many physicists in various fields such as gauge field theory, grand unified theory, particle physics and cosmology [15–20]. In the similar fashion, it is also interesting to consider the gravitomagnetic charge, which is the source of gravitomagnetic field just as mass (gravitoelectric charge) is the source of gravitoelectric field (Newtonian gravitational field). In this sense, gravitomagnetic charge is also termed *dual mass*. It should be noted that the concept of mass is of no significance for the gravitomagnetic charge and it is therefore of interest to investigate the relativistic dynamics and gravitational effects of this topological *dual mass*.

The paper is organized as follows: The gravitational field equation of *gravitomagnetic matter* is derived in Sec. II, the form of field equation in the weak-field approximation is given in Sec. III and the exact static spherically symmetric solution is obtained in Sec. IV. In Sec. V, two related problems, *i.e.*, the geometric phase factor possessed by a photon propagating in the gravitomagnetic field, and a potential interpretation, by means of the mechanism of gravitational Meissner effect, of the anomalous acceleration acting on Pioneer spacecrafts [21] are briefly proposed. In Sec. VI, the author concludes with some remarks.

II. GRAVITATIONAL FIELD EQUATION OF GRAVITOMAGNETIC MATTER

In order to obtain the gravitational field equation of gravitomagnetic charge, we should construct the dual Einstein's tensor. By using the variational principle, we can obtain the following equation

$$\delta \int_{\Omega} \sqrt{-g} \tilde{R} d\Omega = \int_{\Omega} \sqrt{-g} \tilde{G}_{\mu\nu} \delta g^{\mu\nu} d\Omega \quad (1)$$

with

$$\tilde{R} = g^{\sigma\tau} \tilde{R}_{\sigma\tau}, \quad \tilde{R}_{\nu\gamma} = g^{\mu\delta} (\epsilon_{\mu\nu}^{\alpha\beta} R_{\gamma\delta\alpha\beta} + \epsilon_{\gamma\delta}^{\alpha\beta} R_{\mu\nu\alpha\beta}) \quad (2)$$

and

$$\tilde{G}_{\mu\nu} = \epsilon_{\mu}^{\lambda\sigma\tau} R_{\nu\lambda\sigma\tau} - \epsilon_{\nu}^{\lambda\sigma\tau} R_{\mu\lambda\sigma\tau}, \quad (3)$$

where¹ $\epsilon_{\mu}^{\lambda\sigma\tau} = g_{\mu\nu} \epsilon^{\nu\lambda\sigma\tau}$ with $\epsilon^{\nu\lambda\sigma\tau}$ being the completely antisymmetric Levi-Civita tensor, and the dual scalar curvature \tilde{R} is assumed to be the Lagrangian density of the interaction of metric fields with gravitomagnetic matter. Since the dual Einstein's tensor, $\tilde{G}_{\mu\nu}$, is an antisymmetric tensor, we construct the following antisymmetric tensor for the Fermi field

$$K_{\mu\nu} = i\bar{\psi}(\gamma_{\mu}\partial_{\nu} - \gamma_{\nu}\partial_{\mu})\psi, \quad H_{\mu\nu} = \epsilon_{\mu\nu}^{\alpha\beta} K_{\alpha\beta} \quad (4)$$

and regard them as the source tensors in the field equation of gravitomagnetic charge, where γ_{μ} 's are general Dirac matrices with respect to x^{μ} and satisfy $\gamma_{\mu}\gamma_{\nu} + \gamma_{\nu}\gamma_{\mu} = 2g_{\mu\nu}$. Then the field equation of gravitational field produced by the gravitomagnetic charge may be given as follows²

$$\tilde{G}_{\mu\nu} = \kappa_1 K_{\mu\nu} + \kappa_2 H_{\mu\nu} \quad (5)$$

with κ_1, κ_2 being the coupling coefficients between *gravitomagnetic matter* and gravity. It should be noted that $\tilde{G}_{\mu\nu} \equiv 0$ in the absence of gravitomagnetic matter since no singularities associated with topological charge exist in the metric functions and therefore Ricci identity still holds. However, once the metric functions possess non-analytic properties in the presence of gravitomagnetic matter (should such exist), $\tilde{G}_{\mu\nu}$, is no longer vanishing due to the

¹When it comes to the description problem of fields arising from the dual charges, the dual tensor $\tilde{G}_{\mu\nu}$ in gravity theory is analogous to the dual field tensor $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ in electrodynamics.

²One of the advantages of this field equation is that it does not introduce extra tensor potentials when allowing for gravitomagnetic monopole densities and currents. This fact is in analogy with that in electrodynamics, where the equation $\partial_{\nu}\tilde{F}^{\mu\nu} = J_{\text{M}}^{\mu}$ governs the motion of electromagnetic fields produced by magnetic monopoles.

violation of Ricci identity. Additionally, further investigation shows that the *cosmological term* of Fermi field in Eq. (5) can be written as the linear combination of the antisymmetric tensors $i\bar{\psi}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\psi$ and $i\epsilon_{\mu\nu}^{\alpha\beta}\bar{\psi}(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha)\psi$.

It is believed that there would exist formation (and creation) mechanism of gravitomagnetic charge in the gravitational interaction, just as some prevalent theories [18] provide the theoretical mechanism of existence of magnetic monopole in various gauge interactions. Magnetic monopole in electrodynamics and gauge field theory has been discussed and sought after for decades, and the existence of the 't Hooft-Polyakov monopole solutions [18,22,23] has spurred new interest of both theorists and experimentalists. Similar to magnetic monopole, gravitomagnetic charge is believed to give rise to such situations. If it is indeed present in universe, it will also lead to significant consequences in astrophysics and cosmology. We emphasize that although it is the classical solution to the field equation as discussed above, this kind of topological gravitomagnetic monopoles may arise not as fundamental entities in gravity theory.

III. LOW-MOTION WEAK-FIELD APPROXIMATION

In what follows the low-motion weak-field approximation is applied to the following gravitational field equation of *gravitomagnetic matter*

$$\tilde{G}^{\mu\nu} = S^{\mu\nu} \quad (6)$$

with the source tensor $S^{\mu\nu} = \kappa_1 K^{\mu\nu} + \kappa_2 H^{\mu\nu}$. First we consider $\tilde{G}^{01} = \epsilon^{0\alpha\beta\gamma} R^1{}_{\alpha\beta\gamma} - \epsilon^{1\alpha\beta\gamma} R^0{}_{\alpha\beta\gamma}$ by the linear approximation. The following expressions may be given

$$\begin{aligned} -\epsilon^{1\alpha\beta\gamma} R^0{}_{\alpha\beta\gamma} &\simeq 2(R_{0302} + R_{0230}), \\ 2R_{0302} &\simeq \frac{\partial^2 g_{02}}{\partial x^3 \partial x^0} + \frac{\partial^2 g_{30}}{\partial x^0 \partial x^2} - \frac{\partial^2 g_{00}}{\partial x^3 \partial x^2} - \frac{\partial^2 g_{32}}{\partial x^0 \partial x^0}, \\ 2R_{0230} &\simeq \frac{\partial^2 g_{00}}{\partial x^2 \partial x^3} + \frac{\partial^2 g_{23}}{\partial x^0 \partial x^0} - \frac{\partial^2 g_{03}}{\partial x^2 \partial x^0} - \frac{\partial^2 g_{02}}{\partial x^0 \partial x^3}, \end{aligned} \quad (7)$$

where the nonlinear terms (the products of two Christoffel symbols) are ignored and use is made of $\epsilon^{1023} = \epsilon^{1302} = \epsilon^{1230} \simeq -1$ and $R^0{}_{023} \simeq R_{0023} = 0$, $R^0{}_{302} \simeq R_{0302}$, $R^0{}_{230} \simeq R_{0230}$. If metric functions $g_{\mu\nu}$ are analytic, then $2(R_{0302} + R_{0230})$ is therefore vanishing. But once gravitomagnetic charge is present in spacetime and thus the metric functions possess the singularities, this result does not hold. Taking the gravitomagnetic vector potential $\mathbf{g} = (g^{01}, g^{02}, g^{03})$, $-\mathbf{g} = (g_{01}, g_{02}, g_{03})$, one can arrive at

$$-\epsilon^{1\alpha\beta\gamma} R^0{}_{\alpha\beta\gamma} = -\frac{\partial}{\partial x^0}(\nabla \times \mathbf{g})_1 - \nabla \times (\nabla g^{00} - \frac{\partial \mathbf{g}}{\partial x^0})_1. \quad (8)$$

When we utilize the linear approximation for the field equation of *dual matter* (gravitomagnetic matter), we are concerned only with the space-time derivatives of gravitational potentials $g^\mu = (\frac{g^{00}-1}{2}, g^{01}, g^{02}, g^{03})$ rather than that of g_{ii} and g_{ij} with $i, j = 1, 2, 3$, since the latter is either the analytic functions or the small terms. This, therefore, implies that the contribution of $\epsilon^{0\alpha\beta\gamma} R^1{}_{\alpha\beta\gamma}$ to \tilde{G}^{01} vanishes. Eq. (8) is readily generalized to the three-dimensional case, and the combination of Eq. (6) and Eq. (8) yields

$$\nabla \times (\nabla g^{00} - \frac{\partial}{\partial x^0} \mathbf{g}) = -\frac{\partial}{\partial x^0}(\nabla \times \mathbf{g}) + \vec{S}, \quad (9)$$

where \vec{S} is defined to be S^{i0} ($i = 1, 2, 3$). It is apparently seen that Eq. (9) is exactly analogous to the Faraday's law of electromagnetic induction in the presence of current density of magnetic monopole in electrodynamics. This, therefore, implies that Eq. (6) is indeed the field equation of gravitation of gravitomagnetic matter.

It is also of interest to discuss the motion of gravitomagnetic monopole in curved spacetime. Although Ricci identity is violated due to the non-analytic properties caused by the gravitomagnetic charge, Bianchi identity still holds in the presence of gravitomagnetic charge. It follows that the covariant divergence of $\tilde{G}^{\mu\nu}$ vanishes, namely,

$$\tilde{G}^{\mu\nu}{}_{;\nu} = 0. \quad (10)$$

Then in terms of the following field equation

$$\tilde{G}^{\mu\nu} = S^{\mu\nu} \quad (11)$$

with the antisymmetric source tensor of gravitomagnetic matter $S^{\mu\nu}$ being $\kappa_1 K^{\mu\nu} + \kappa_2 H^{\mu\nu}$, one can arrive at

$$S^{\mu\nu}{}_{;\nu} = 0 \quad (12)$$

which may be regarded as the equation of motion of gravitomagnetic charge in the curved spacetime. It is useful to obtain the low-motion and weak-field-approximation form of Eq. (12), which enables us to guarantee that Eq. (12) is the equation of motion of gravitomagnetic monopole indeed.

The general Dirac matrices in the weak-field approximation may be obtained via the relations $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}$ and the results are given by

$$\gamma^0 = (1 + g^0)\beta, \quad \gamma^i = g^i\beta + \gamma_M^i, \quad (13)$$

where $i = 1, 2, 3$; $\beta = \gamma_M^0$, γ_M^0 and γ_M^i are the constant Dirac matrices in the flat Minkowski spacetime. Note that the gravitoelectric potential is defined to be $g^0 = \frac{g^{00}-1}{2}$, and gravitomagnetic vector potentials are $g^i = g^{0i}$ ($i = 1, 2, 3$). In the framework of dynamics of point-like particle, the source tensor is therefore rewritten as

$$S^{\mu\nu} = \rho [\kappa_1 (g^\mu U^\nu - g^\nu U^\mu) + \kappa_2 \epsilon^{\mu\nu\alpha\beta} (g_\alpha U_\beta - g_\beta U_\alpha)], \quad (14)$$

where ρ denotes the density of gravitomagnetic matter. It follows from Eq. (12) and Eq. (14) that there exists the gravitational Lorentz force density in the expression for the force acting on the gravitomagnetic charge, namely, the equation of motion of gravitomagnetic charge is of the form

$$\frac{\partial}{\partial x^0} \mathbf{v} = [\nabla \times \mathbf{g} - \mathbf{v} \times (\nabla g^0 - \frac{\partial}{\partial x^0} \mathbf{g})], \quad (15)$$

where some small terms are ignored and the relation, $\kappa_1 g^0 = 2\kappa_2$, between the coupling coefficients, κ_1 and κ_2 is assumed (one may be referred to the Appendix for the consideration why this assumption is reasonable). Note, however, that the relation of the two coupling parameters suggested above holds only when weak-field approximation is employed (see Appendix for more detailed analysis). This connection between κ_1 and κ_2 gives us a helpful insight into the generally covariant relation between them.

In view of what has been discussed, it can be seen that, in the weak gravitational field, the gravitomagnetic charge behaves like the magnetic charge. This, therefore, implies that gravitomagnetic charge proposed above is the gravitational analogue of magnetic charge in electrodynamics.

IV. EXACT SOLUTION OF STATIC SPHERICALLY SYMMETRIC GRAVITOMAGNETIC FIELD

A static spherically symmetric solution is exactly obtained by supposing that when a point-like gravitomagnetic charge is fixed at the origin of the spherical coordinate system, the exterior spacetime interval is of the form

$$ds^2 = (dx^0)^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) + 2g_{0\varphi}(\theta)dx^0 d\varphi, \quad (16)$$

where the gravitomagnetic potential $g_{0\varphi}$ is assumed to be the function with respect only to θ .

Thus we obtain all the Christoffel symbols with non-vanishing values as follows:

$$\Gamma_{0,\varphi\theta} = \Gamma_{0,\theta\varphi} = \frac{1}{2} \frac{\partial g_{0\varphi}}{\partial \theta}, \quad \Gamma_{\varphi,\theta\theta} = \Gamma_{\varphi,\theta\theta} = \frac{1}{2} \frac{\partial g_{0\varphi}}{\partial \theta}, \quad \Gamma_{\theta,0\varphi} = \Gamma_{\theta,\varphi 0} = -\frac{1}{2} \frac{\partial g_{0\varphi}}{\partial \theta}. \quad (17)$$

Since the field equation of gravitomagnetic matter is the antisymmetric equation, we might as well take into account a simple case of the following equation

$$\epsilon^{0\alpha\beta\gamma} R^0{}_{\alpha\beta\gamma} = M\delta(x^i) \quad (18)$$

with M being the parameter associated with the coupling parameters and gravitomagnetic charge. It is therefore apparent that Eq. (18) agrees with Eq. (11). Hence, the solution of the former equation also satisfies the latter. For the reason of the completely antisymmetric property of the Levi-Civita tensor, the contravariant indices α, β, γ should be respectively taken to be x, y, z of three-dimensional space coordinate, namely, we have

$$\epsilon^{0\alpha\beta\gamma} R^0{}_{\alpha\beta\gamma} = 2\epsilon^{0xyz} (R^0{}_{xyz} + R^0{}_{zyx} + R^0{}_{yxz}). \quad (19)$$

There exist the products of two Christoffel symbols, *i.e.*, $g^{\sigma\tau}(\Gamma_{\tau,\alpha\gamma}\Gamma_{\lambda,\sigma\beta} - \Gamma_{\tau,\alpha\beta}\Gamma_{\lambda,\sigma\gamma})$ in the definition of the Riemann curvature, $R_{\lambda\alpha\beta\gamma}$. Apparently, the products of two Christoffel symbols (the nonlinear terms of field equation) contain the total indices of three-dimensional space coordinate (namely, these indices are taken the permutations of r, θ, φ) and therefore vanish, in the light of the fact that the Christoffel symbol with index r is vanishing in terms of Eq. (17).

In view of the above discussion, one can conclude that Eq. (18) can be exactly reduced to a linear equation. It is easily verified that $R_{\lambda\alpha\beta\gamma}$ ($\lambda = r, \theta, \varphi$) vanishes with the help of the linear expression for $R_{\lambda\alpha\beta\gamma}$ given by $R_{\lambda\alpha\beta\gamma} = \frac{1}{2}(\frac{\partial^2 g_{\lambda\gamma}}{\partial x^\alpha \partial x^\beta} + \frac{\partial^2 g_{\alpha\beta}}{\partial x^\lambda \partial x^\gamma} - \frac{\partial^2 g_{\lambda\beta}}{\partial x^\alpha \partial x^\gamma} - \frac{\partial^2 g_{\alpha\gamma}}{\partial x^\lambda \partial x^\beta})$ and the linear element expressed by Eq. (16). We thus obtain that $R^0_{\alpha\beta\gamma} = g^{00}R_{0\alpha\beta\gamma}$. By the aid of the following expression

$$R_{0\alpha\beta\gamma} = \frac{1}{2} \frac{\partial}{\partial x^\alpha} (\frac{\partial g_{0\gamma}}{\partial x^\beta} - \frac{\partial g_{0\beta}}{\partial x^\gamma}), \quad (20)$$

one can arrive at

$$\epsilon^{0\alpha\beta\gamma} R^0_{\alpha\beta\gamma} = -\frac{g^{00}}{\sqrt{-g}} \nabla \cdot (\nabla \times \mathbf{g}), \quad (21)$$

where the gravitomagnetic vector potentials, \mathbf{g} , are defined to be $\mathbf{g} = (-g_{0x}, -g_{0y}, -g_{0z})$. Substitution of Eq. (21) into Eq. (18) yields

$$\nabla \cdot (\nabla \times \mathbf{g}) = -\frac{\sqrt{-g}}{g^{00}} M \delta(x^i). \quad (22)$$

Note that Eq. (22) is the exact static gravitational field equation of gravitomagnetic matter derived from Eq. (11), where use is made of the expression (16) for the spacetime interval.

It follows from Eq. (22) that the static spherically symmetric solution is given as follows

$$2g_{0\varphi} dx^0 d\varphi = \mp \frac{2c}{4\pi} \cdot \frac{1 \pm \cos \theta}{r \sin \theta} \cdot r \sin \theta dx^0 d\varphi, \quad (23)$$

where c is defined to be determined by the metric functions of the origin of the spherical coordinate system, *i.e.*, $c = -M(\frac{\sqrt{-g}}{g^{00}})_{\text{origin}}$. Further calculation yields

$$(g^{\mu\nu}) = \begin{pmatrix} \frac{r^2 \sin^2 \theta}{r^2 \sin^2 \theta + g_{0\varphi}^2} & 0 & 0 & \frac{g_{0\varphi}}{r^2 \sin^2 \theta + g_{0\varphi}^2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^{-2} & 0 \\ \frac{g_{0\varphi}}{r^2 \sin^2 \theta + g_{0\varphi}^2} & 0 & 0 & \frac{-1}{r^2 \sin^2 \theta + g_{0\varphi}^2} \end{pmatrix} \quad (24)$$

which is the inverse matrix of the metric $(g_{\mu\nu})$ and we thus obtain the contravariant metric $g^{\mu\nu}$. The gravitomagnetic field strength takes the form

$$B_g^i = \frac{c}{4\pi} \frac{x^i}{r^3}, \quad (25)$$

provided that use is made of $\mathbf{B}_g = \nabla \times \mathbf{A}$ with $\mathbf{A} = (0, 0, \mp \frac{2c}{4\pi} \frac{1 \pm \cos \theta}{r \sin \theta})$ expressed in spherical coordinate system.

From what has been discussed above, similar to the magnetic charge in electrodynamics, gravitomagnetic charge is a kind of topological charge that is the duality of mass of matter. In this sense, gravitomagnetic charge is also called dual mass. From the point of view of the classical field equation, matter may be classified into two categories: gravitomagnetic matter and gravitoelectric matter, of which the field equation of the latter is Einstein's equation of general relativity. Due to their different gravitational features, the concept of mass is of no significance for the gravitomagnetic matter.

V. TWO RELATED PROBLEMS

(1) It is worthwhile to take into account the motion of photon in gravitomagnetic fields. Consider a weak gravitomagnetic field where the adiabatic approximation can be applicable to the motion of a photon, a conclusion can be

drawn that the eigenvalue of the helicity $\frac{\mathbf{k}(t)}{k} \cdot \mathbf{J}$ of the photon is conserved in motion and the helicity operator $\frac{\mathbf{k}(t)}{k} \cdot \mathbf{J}$ is an invariant in terms of the invariant equation (*i.e.*, the Liouville-Von Neumann equation) in Lewis-Riesenfeld theory [24]

$$\frac{\partial I(t)}{\partial t} + \frac{1}{i}[I(t), H(t)] = 0, \quad (26)$$

where the invariant $I(t) = \frac{\mathbf{k}(t)}{k} \cdot \mathbf{J}$. From Eq. (26), simple calculation yields

$$H(t) = \frac{\mathbf{k} \times \frac{d}{dt}\mathbf{k}}{k^2} \cdot \mathbf{J}, \quad (27)$$

which is considered an effective Hamiltonian governing the motion of photon in gravitomagnetic fields. Hence, the equation of motion of a photon in gravitomagnetic fields is written

$$\frac{\mathbf{k} \times \frac{d}{dt}\mathbf{k}}{k^2} = \mathbf{B}_g, \quad (28)$$

where the gravitomagnetic field strength \mathbf{B}_g is determined by the field equations of gravitation such as Eq. (11) and Einstein's equation of general relativity. It follows from the geodesic equation in the weak-field approximation that the acceleration due to gravitational Lorentz force is

$$\frac{1}{k} \frac{d}{dt}\mathbf{k} = -\frac{\mathbf{k}}{k} \times (\nabla \times \vec{g}) \quad (29)$$

with $\frac{\mathbf{k}}{k}$ being the velocity vector of the photon, and the gravitomagnetic field strength is $\mathbf{B}_g = \nabla \times \mathbf{g}$, where $\mathbf{g} = (g^{01}, g^{02}, g^{03})$. Substitution of Eq. (28) into Eq. (29) yields

$$\frac{1}{k} \frac{d}{dt}\mathbf{k} = -\frac{\mathbf{k}}{k} \times \left(\frac{\mathbf{k} \times \frac{d}{dt}\mathbf{k}}{k^2} \right). \quad (30)$$

Since

$$\mathbf{k} \cdot \mathbf{k} = k^2, \quad \mathbf{k} \cdot \frac{d}{dt}\mathbf{k} = 0, \quad (31)$$

Eq. (30) is proved to be an identity. This, therefore, implies that Eq. (28) is truly the equation of motion of a photon in gravitomagnetic fields.

For the time-dependent gravitomagnetic fields, similar to the case of the photon propagating inside the noncoplanarly curved optical fiber that is wound smoothly on a large enough diameter [25–27], the photon propagating in the gravitomagnetic field would also give rise to a geometric phase, which can be calculated by making use of the Lewis-Riesenfeld invariant theory [24] and the invariant-related unitary transformation formulation [28], and the result is

$$\phi_{\pm}^{(g)} = \pm \int_0^t \dot{\gamma}(t') [1 - \cos \lambda(t')] dt' \quad (32)$$

with \pm corresponding to the eigenvalue $\sigma = \pm 1$ of the helicity $\frac{\mathbf{k}(t)}{k} \cdot \mathbf{J}$ of the photon. The time-dependent parameters, γ and λ , are so defined that $\frac{\mathbf{k}(t)}{k} = (\sin \lambda(t) \cos \gamma(t), \sin \lambda(t) \sin \gamma(t), \cos \lambda(t))$. Differing from the dynamical phase that is related to the energy, frequency or velocity of a particle or a quantum system, geometric phase is dependent only on the geometric nature of the pathway along which the system evolves. For the case of adiabatic process where λ does not explicitly involve time t , Eq. (32) is reduced to

$$\Delta\theta = \pm 2\pi(1 - \cos \lambda) \quad (33)$$

in one cycle over the parameter space of the helicity $\frac{\mathbf{k}(t)}{k} \cdot \mathbf{J}$. It follows from Eq. (14) that $2\pi(1 - \cos \lambda)$ is the expression for a solid angle in $\mathbf{k}(t)$ space, which presents the geometric properties of time evolution of the interaction between the gravitomagnetic field and photon spin (gravitomagnetic moment).

(2) Taking the effects of gravitomagnetic fields and gravitomagnetic charges into consideration is believed to be of essential significance in resolving some problems and paradoxes. An illustrative example that would be briefly

discussed in what follows may be regarded as an application of gravitomagnetic fields (matter) to the cosmological constant problem. The gravitational analogue of Meissner effect in superconductivity is gravitational Meissner effect. Due to the conservation of momentum, mass-current density is conserved before and after the scatterings of particles in perfect fluid, which is analogous to the superconductivity of superconducting electrons in superconductors cooled below T_c . Since gravitational field equation of linear approximation is similar to the Maxwell's equation in electrodynamics, one can predict that gravitational Meissner effect arises also in perfect fluid. The author think that the investigation of both the effect of gravitomagnetic matter and gravitational Meissner effect may provide us with a valuable insight into the problem of cosmological constant and vacuum gravity [29–32]. Given that the vacuum matter is perfect fluid, the gravitoelectric field (Newtonian gravitational field) produced by the gravitoelectric charge (mass) of the vacuum quantum fluctuations is exactly cancelled by the gravitoelectric field due to the induced current of the gravitomagnetic charge of the vacuum quantum fluctuations; the gravitomagnetic field produced by the gravitomagnetic charge (dual mass) of the vacuum quantum fluctuations is exactly cancelled by the gravitomagnetic field due to the induced current of the gravitoelectric charge (mass current) of the vacuum quantum fluctuations. Thus, at least in the framework of weak-field approximation, the extreme space-time curvature of vacuum caused by its large energy density does not arise, and the gravitational effects of cosmological constant³ is eliminated by the contributions of the gravitomagnetic charge (dual mass).

Additionally, in 1998, Anderson *et al.* reported that, by ruling out a number of potential causes, radio metric data from the Pioneer 10/11, Galileo and Ulysses spacecraft indicate an apparent anomalous, constant, acceleration acting on the spacecraft with a magnitude $\sim 8.5 \times 10^{-8} \text{cm/s}^2$ directed towards the Sun [21]. Is it the effects of dark matter or a modification of gravity? Unfortunately, neither easily works. By taking the cosmic mass, $M = 10^{53} \text{kg}$, and cosmic scale, $R = 10^{26} \text{m}$, calculation shows that this acceleration is just equal to the value of field strength on the cosmic boundary due to the total cosmic mass. This fact leads us to consider a theoretical mechanism to interpret this anomalous phenomenon. The author favors that the gravitational Meissner effect may serve as a possible interpretation. Here we give a rough analysis, which contains only the most important features rather than the precise details of this theoretical explanation. Parallel to London's electrodynamics of superconductivity, it shows that gravitational field may give rise to an *effective rest mass* $m_g = \frac{\hbar}{c^2} \sqrt{8\pi G \rho_m}$ due to the self-induced charge current [33], where ρ_m is the mass density of the universe. Then one can obtain that $\frac{\hbar}{m_g c} \simeq 10^{26} \text{m}$ that approximately equals R , where the mass density of the universe is taken to be $\rho_m = 0.3\rho_c$ [34] with $\rho_c \simeq 2 \times 10^{-26} \text{kg/m}^3$ being the critical mass density. An added constant acceleration, a , may result from the Yukawa potential and can be written as⁴

$$a = \frac{GM}{2} \left(\frac{m_g c}{\hbar} \right)^2 \simeq \frac{GM}{R^2}. \quad (34)$$

Note, however, that it is an acceleration of repulsive force directed, roughly speaking, from the center of the universe. By analyzing the NASA's Viking ranging data, Anderson, Laing, Lau *et al.* concluded that the anomalous acceleration does not act on the body of large mass such as the Earth and Mars. If gravitational Meissner effect only affected the gravitating body of large mass or large scale rather than spacecraft (perhaps the reason lies in that small-mass flow cannot serve as self-induced charge current, which deserves to be further investigated), then seen from the Sun or Earth, there exists an added attractive force acting on the spacecraft. This added force give rise to an anomalous, constant, acceleration directed towards the Sun or Earth. It should be emphasized that the above theoretical theme is only a potential interpretation of this anomalous gravitational phenomenon. However, the sole reason that the above resolution of the anomalous acceleration is somewhat satisfactory lies in that no adjustable parameters exist in this theoretical framework. It is one of the most important advantages in the above mechanism of gravitational Meissner effect, compared with some possible theories of modification of gravity [35], which are always involving several parameters that cannot be determined by theory itself. These theories of modification of gravity were applied to the problem of the anomalous acceleration but could not calculate the value of the anomalous acceleration.

³For the more present considerations regarding the cosmological constant problem, see, for example, hep-ph/0002297 (by E. Witten) and astro-ph/0005265 (by S. Weinberg).

⁴It may also be calculated as follows: $a = \frac{GM}{2} \left(\frac{m_g c}{\hbar} \right)^2 = \frac{GM}{c^2} (4\pi \rho_m G) = \frac{GM}{c^2 R} \frac{G(4\pi R^3 \rho_m)}{R^2} \simeq \frac{GM}{R^2}$, where use is made of $\frac{GM}{c^2 R} \simeq \mathcal{O}(1)$, $4\pi R^3 \rho_m \simeq M$, which holds when the approximate estimation is performed.

In summary, in the present paper the author investigates some geometric effects, gravitational analogues of electromagnetic phenomena, and the field equation of gravitomagnetic matter (dual matter) as well as its static spherically symmetric solution. Differing from the symmetric property (with respect to the indices of tensors) of gravitational field equation of gravitoelectric matter, the field equation of gravitomagnetic matter possesses the antisymmetric property. This, therefore, implies that the number of the non-analytic metric functions is no more than 6. Although we have no observational evidences for the existence of gravitomagnetic charge, it is still of essential significance to investigate the gravity theory of the topological dual mass⁵.

Some physically interesting problems associated with gravitomagnetic fields are proposed, of which the most interesting investigation is the potential solution to the anomalous acceleration acting on Pioneer spacecrafts by means of the mechanism of gravitational Meissner effect. The theoretical resolution of this problem is not very definite at present, for it cannot account for why the gravitational Meissner effect does not explicitly affect the body of small mass. This curiosity deserves further considerations.

Since with foreseeable improvements in detecting and measuring technology, it is possible for us to investigate quantum mechanics in weak-gravitational fields, the above effects and phenomena deserve further detailed investigations. Work in this field is under consideration and will be published elsewhere.

APPENDIX

It is necessary to analyze the relation between the coupling parameters, κ_1 and κ_2 , under the low-motion and weak-field approximation. Substitution of the expression (14) for $S^{\mu\nu}$ into Eq. (12), where

$$S^{\mu\nu}{}_{;\nu} = \frac{\partial S^{\mu\nu}}{\partial x^\nu} + \frac{1}{2} S^{\mu\lambda} g^{\sigma\tau} \frac{\partial g_{\sigma\tau}}{\partial x^\lambda}, \quad (\text{A1})$$

yields

$$\begin{aligned} \kappa_1 g^0 \frac{\partial}{\partial x^0} \mathbf{v} &= 2\kappa_2 [\nabla \times \mathbf{g} - \mathbf{v} \times (\nabla g^0 - \frac{\partial}{\partial x^0} \mathbf{g})] - 2\kappa_2 \mathbf{g} \times (\frac{\partial}{\partial x^0} \mathbf{v} + \nabla g^0) \\ &+ \kappa_1 g^0 \frac{\partial}{\partial x^0} \mathbf{g} - 2\kappa_2 g^0 (\nabla g^0 \times \mathbf{v}) \end{aligned} \quad (\text{A2})$$

with \mathbf{v} being the velocity of the tested gravitomagnetic monopole. It is apparent that $\nabla \times \mathbf{g} - \mathbf{v} \times (\nabla g^0 - \frac{\partial}{\partial x^0} \mathbf{g})$ is the expression associated with gravitational Lorentz force density. Note that in Eq. (A2), κ_1, κ_2 are considered coupling constants. However, further analysis shows that at least one of them is not a constant, and if the relation

$$\kappa_1 g^0 = 2\kappa_2 \quad (\text{A3})$$

between them is assumed, then Eq. (A2) may be rewritten as

$$\frac{\partial}{\partial x^0} \mathbf{v} = [\nabla \times \mathbf{g} - \mathbf{v} \times (\nabla g^0 - \frac{\partial}{\partial x^0} \mathbf{g})], \quad (\text{A4})$$

where we temporarily ignore the second and third terms, which can be considered the small terms, on the right-hand side of Eq. (A2) and the derivative term of coupling coefficients with respect to space-time coordinates x^μ . It is well known that the form of Eq. (A4) is the equation of motion of a particle acted upon by the Lorentz force. Hence, Eq. (12) is believed to be the generally relativistic equation of motion of gravitomagnetic monopole in the Riemann space-time.

⁵Many investigations concerning the gravitomagnetic monopole and its quantization as well as the related effects and phenomena such as the gravitational lensing and the orbits of motion of matter in NUT space were performed. Readers can be referred to [Rev. Mod. Phys. **70**(2), 427-445(1998)](see also in gr-qc/9612049) and the references therein. However, we think in these references the gravitational field equation of gravitomagnetic monopole may get less attentions than it deserves.

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